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A Unique Limiting Green's Function for a Class of Singular Boundary Value Problems

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Abstract—The behaviors of the Green's functions for a family of two point boundary value problems are analyzed as the right hand boundary point diverges to ∞. In particular, a unique limiting Green's function is obtained. The sign properties of the unique limiting function are analyzed and it is shown that well-defined boundary conditions are satisfied at ∞.

Keywords—Singular boundary value problems, Green's functions, Right disfocality, Maximum principle.

Let $a \in \mathbb{R}$ be fixed, and let $b-a$ be a positive integer. Let $I_b = \{a, a+1, \dots, b\}$, let $n \geq 1$ be an integer, and let $I_b^l = \{a, a+1, \dots, b+l\}$, $l = 0, 1, \dots, n$. We shall consider the linear, n^{th} order difference operator, P defined by

$$Pu(m) = \Delta^n u(m) + q_0(m)u(m) + \cdots + q_{n-1}(m)u(m+n-1), \qquad m \in I_b, \tag{1}$$

where $\Delta^0 u(m) = u(m)$, $\Delta u(m) = u(m+1) - u(m)$, $\Delta^l u(m) = \Delta(\Delta^{l-1}u(m))$, $l = 2, \dots, n$, and $q_l(m), l = 0, \dots, n-1$, is defined for $m \in I_\infty = \{a, a+1, \dots\}$.

Recall [1] that for a finite or infinite sequence of real numbers, $u = u(a), u(a+1), \dots$, $m = a$ is a *generalized zero* (g.z.) for u if $u(a) = 0$, and $m > a$ is a *generalized zero* (g.z.) for u if $u(m) = 0$, or there is an integer, $l \geq 1$, such that $m - l \geq a$, $(-1)^l u(m-l)u(m) > 0$, and if $l > 1$, then $u(m-l-1) = \cdots = u(m-1) = 0$. Moreover, recall [2,3] that the difference equation, $Pu(m) = 0, m \in I_b$, is *right disfocal* on I_b, if the only solution, u, of $Pu(m) = 0, m \in I_b$, satisfying $\Delta^{l-1}u$ has a generalized zero at $m_l, l = 1, \dots, n$, where $a \leq m_1 \leq \cdots \leq m_n \leq b+1$, is $u \equiv 0$. Throughout this paper, we shall assume that $Pu(m) = 0, m \in I_\infty$, is right disfocal in I_∞.

Let N denote the set of nonnegative integers; let $k \in \{1, \dots, n-1\}$ and define $\Omega \subset N^{n-k}$ by

$$\Omega = \left\{ i = (i_1, \dots, i_{n-k}) : 0 \leq i_1 < \cdots < i_{n-k} \leq n-1 \right\}. \tag{2}$$

For each $b \in I_\infty, b \geq a+k$, and for each $i \in \Omega$, consider homogeneous, two-point boundary conditions of the form

$$\begin{aligned} \Delta^l u(a) &= 0, \qquad l = 0, \dots, k-1; \\ \Delta^l u(b+n-l) &= 0, \qquad l = i_1, \dots, i_{n-k}. \end{aligned} \tag{3(b,i)}$$

Denote the boundary conditions, (3(b,i)), by $T(b,i)u = 0$. If $i = (0, \dots, n-k-1)$, then $T(b,i)u = 0$ represents conjugate boundary conditions [1] ; if $i = (k, \dots, n-1)$, then $T(k,i)u = 0$ represents right focal boundary conditions [2,3].

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For each $i \in \Omega$, and for each $b \in I_\infty$, $b \geq a+k-i_1$, let $G(b,i;m,s)$ denote the Green's function for the boundary value problem (BVP), $Pu(m) = 0$, $m \in I_b$, $T(b,i)u = 0$. Note that $G(b,i;m,s)$ is defined on $I_b^n \times I_b$, if it exists; see [1,4]. Moreover, under the right disfocality assumption, $G(b,i;m,s)$ exists for each $i \in \Omega$, and for each $b \in I_\infty$, $b \geq a+k-i_1$; see [2,5]. Finally, let $\Delta G(b,i;m,s) = G(b,i;m+1,s) - G(b,i;m,s)$ denote the partial difference of G with respect to m; let $\Delta^l G = \Delta(\Delta^{l-1}G)$, $l = 2,\ldots,n$.

Throughout this paper, we shall impose a sign condition on the coefficients, q_l, $l = 0,\ldots,n-1$, in (1); in particular, we shall require

$$(-1)^{n-k} q_l(m) \geq 0, \qquad m \in I_\infty, \quad l = 0,\ldots,n-1. \tag{4}$$

The purpose of this paper is to study the asymptotic properties of $G(b,i;m,s)$ as $b \to \infty$, for each $i \in \Omega$. In particular, we shall show that if $i \in \Omega$, then $\lim_{b\to\infty} G(b,i;m,s) = \mathbb{G}(i;m,s)$ is a well-defined function of known sign on $I_\infty \times I_\infty$. Moreover, we shall show that for $i,j \in \Omega$, $\mathbb{G}(i;m,s) = \mathbb{G}(j;m,s) = \mathbb{G}(m,s)$. Finally, we shall characterize the boundary conditions that the unique limiting function, $\mathbb{G}(m,s)$, satisfies at $m = \infty$.

The concept of a unique limiting Green's function is well developed in the limit point case for singular ordinary differential equations [6]. However, our study here is motivated by the study by Elias [7] for singular two-term ordinary differential equations; in particular, to obtain that the unique limiting function is of known sign, and to characterize the boundary conditions at ∞, is beyond the theory of the limit point case for singular ordinary differential equations. We point out that Elias [7] assumes precisely the opposite sign condition as in (4). Hence, the techniques developed in this paper are not analogous to those developed by Elias [7].

Define a partial order on Ω as follows: for $i = (i_1,\ldots,i_{n-k})$, $j = (j_1,\ldots, j_{n-k}) \in \Omega$, we shall say that $i \leq j$ if, and only if, $i_l \leq j_l$, $l = 1,\ldots,n-k$. Moreover, $i < j$ if $i \leq j$, and $i \neq j$.

We now state, without proof, two preliminary theorems and a preliminary lemma which were obtained by Eloe and Peil [5].

THEOREM 1. *Assume that $Pu(m) = 0$, $m \in I_\infty$, is right disfocal on I_∞. Let $i = (i_1,\ldots, i_1+n-k-1)$, $i_1 \leq k-1$, and let $b \in I_\infty$, $b > a+k$. Then*

$$(-1)^{n-k}\Delta^{i_1} G(b,i;m,s) > 0, \qquad (m,s) \in \{a+k-i_1,\ldots,b+k-i_1\} \times I_b.$$

THEOREM 2. *Assume that $Pu(m) = 0$, $m \in I_\infty$, is right disfocal on I_∞. Let $i,j \in \Omega$, $i < j$, $b \in I_\infty$, $b \geq a+k$. Then*

$$(-1)^{n-k}\Delta^{i_1} G(b,j;m,s) > (-1)^{n-k}\Delta^{i_1} G(b,i;m,s), \tag{5}$$

where (5) holds for $(m,s) \in \{a+k-i_1,\ldots,b+n-i_1-\alpha\} \times I_b$, where $\alpha = 0$ if $j_1 > i_1$, or $\alpha > 0$ is such that $j_\alpha = i_1 + (\alpha-1)$, $j_{\alpha+1} > i_1 + \alpha$.

REMARK. It follows from Theorems 1 and 2, and noting the boundary conditions, (3(b,i)), that under the hypotheses of Theorem 2, then

$$(-1)^{n-k}\Delta^k G(b,j;a,s) > (-1)^{n-k}\Delta^k G(b,i;a,s) > 0.$$

LEMMA 3. *Assume that $Pu(m) = 0$, $m \in I_\infty$, is right disfocal on I_∞. Let $i_1,\ldots,i_{n-k-1}$ be integers satisfying $0 \leq i_1 < \cdots < i_{n-k-1} \leq n-1$. Let u be a nontrivial solution of $Pu = 0$, $m \in I_b$, satisfying the $n-1$ boundary conditions,*

$$\begin{aligned} \Delta^l u(a) &= 0, \qquad l = 0,\ldots,k-1; \\ \Delta^l u(b+n-l) &= 0, \qquad l = i_1,\ldots,i_{n-k-1}. \end{aligned}$$

Set $M = \max\{l : l \in \{0,\dots,n-1\},\ \Delta^l u(b+n-l) \neq 0\}$. *Then* $M \geq k$, *and*

(i) $\Delta^l u$ *does not have a g.z. at* $b+n-l$, $l \in \{0,\dots,n-1\} \setminus \{i_1,\dots,i_{n-k-1}\}$;

(ii) $\Delta^l u$ *does not have a g.z. at* $m \in \{a+k-l,\dots,b+n-l-\gamma_l\}$, *for* $l = 0,\dots,\min\{k,i_1\}$, *where* $\gamma_l = \min\{\nu - 1 : \Delta^\nu u(b+n-\nu) \neq 0$ *for* $\nu \in \{l,\dots,n-1\}\}$ *or* $\gamma_l = n-l$, *if* $\Delta^\nu u(b+n-\nu) = 0$ *for* $\nu \in \{l,\dots,n-1\}$;

(iii) *if* $l \in \{i_1,\dots,i_{n-k-1}\}$, *such that* $l \leq M-1$, *then*

$$\big(-1\big)^\nu \Delta^l u(b+n-l-\gamma_l)\, \Delta^{l+\nu} u(b+n-l-\gamma_l) > 0, \qquad \nu = 1,\dots,\gamma_l;$$

(iv) *if* $l \in \{0,\dots,M-1\} \setminus \{i_1,\dots,i_{n-k-1}\}$, *then*

$$\Delta^l u(b+n-l-\nu)\, \Delta^{l+1} u(b+n-l-\gamma_{l+1}-1) > 0,$$

for $\nu = 0,\dots,\gamma_{l+1}$, *and if* $\gamma_{l+1} > 0$,

$$\big(-1\big)^\nu \Delta^l u(b+n-l)\, \Delta^{l+\nu+1} u(b+n-l-\gamma_{l+1}-1) > 0, \qquad \nu = 1,\dots,\gamma_{l+1}.$$

REMARK. In regard to Theorem 2, we shall say that $\big(-1\big)^{n-k}\Delta^{i_1} G(b,i;m,s)$ is monotone increasing as a function of i. In Theorem 5, we shall obtain an analogous comparison theorem for Green's functions as functions of b. In particular, with an additional hypothesis on i_{n-k}, we shall obtain that $\big(-1\big)^{n-k}\Delta^{i_1} G(b,i;m,s)$ is monotone increasing as a function of b. To obtain Theorem 5, we shall need the following lemma.

LEMMA 4. *Assume that* $Pu(m) = 0, m \in I_\infty$, *is right disfocal on* I_∞. *Assume* $i \in \Omega,\ b \geq a+k$. *If* $l \in \{1,\dots,n-1\} \setminus \{i_1,\dots,i_{n-k}\}$, *then*

$$\big(-1\big)^{s(i,l)} \Delta^l G(b,i;b+n-l,s) > 0, \qquad s \in I_b,$$

where $s(i,l)$ *equals the number of components of* i *which exceed* l. *In particular,* $s(i,l)$ *counts the number of differences of* u *of higher order than* l *which the boundary conditions,* $T(b,i)u = 0$, *specify to be zero at the respective right hand endpoint.*

PROOF. The proof is by induction on k. For $k = 1$, first, let $i = (0,1,\dots,n-2)$. By Theorem 1, $\big(-1\big)^{n-1} G(b,i;b+1,s) > 0$. It follows by the algebra of finite differences that $\Delta^{n-1} G(b,i;b\ +\ 1,s) > 0$. In particular, the assertion of the lemma is valid for $k = 1$, where $i = (0,\dots,n-2)$. Let $j \in\ \Omega,\ j \neq i$. Then the components of j exclude precisely one of the integers, $0,\dots,n-2$. Denote the excluded integer by l; $s(j,l) = n-1-l$. Fix $s \in I_b$ and consider $g(m) = G(b,j;m,s) - G(b,i;m,s)$. Then $Pg = 0,\ m \in I_b$ and

$$\begin{aligned} \Delta^p g(a) &= 0, \qquad p = 0,\dots,k-1; \\ \Delta^p g(b+n-p) &= 0, \qquad p \in \{0,\dots,n-k-2\} \setminus \{l\}, \\ \Delta^{n-1} g(b+1) &= -\Delta^{n-1} G(b,i;b+1,s) < 0. \end{aligned}$$

Apply Lemma 3 (iv) with $\gamma_{l+1} = n-l-2$. In particular,

$$\big(-1\big)^{n-1-l} \Delta^l g(b+n-l) = \big(-1\big)^{s(i,l)} \Delta^l G(b,j;b+n-l,s) > 0, \qquad s \in I_b,$$

and the lemma is proved in the case $k = 1$.

Next, assume that Lemma 4 is valid for $k-1 \in \{1,\dots,n-2\}$; we shall establish the validity of Lemma 4 for k. The proof is by induction on $i_1+\cdots+i_{n-k} = S(i)$. Note that $(n-k)(n-k-1)/2 \leq S(i) \leq (n-k)(n+k-1)/2$. First, let $i = \alpha = (0,\dots,n-k-1)$, so that $S(\alpha) = (n-k)(n-k-1)/2$. Since $\big(-1\big)^{n-k} G(b,\alpha;b+k,s) > 0$ and $G(b,\alpha;m,s) = 0,\ m = b+k+1,\dots,b+n$, it follows by the algebra of finite differences that $\Delta^{n-k}G(b,\alpha;b+k,s)>0$. Hence, Lemma 4 is valid for $l=n-k$. Now, let $l\in\{n-k+1,\dots\ n-1\}$. Define $j=(0,\dots,n-k-1,l)$. Note that the inductive hypothesis on $k-1$ applies to j. Again, fix $s\in I_b$ and consider $g(m)=G(b,\alpha;m,s)-G(b,j;m,s)$. Then $Pg=0,\ m\in I_b$ and

$$\Delta^p g(a)=0,\qquad p=0,\dots,k-2;$$
$$\Delta^p g(b+n-p)=0,\qquad p=0,\dots,n-k-1,$$
$$\Delta^{n-k}g(b+k)=\Delta^{n-k}G(b,\alpha;b+k,s)-\Delta^{n-k}G(b,j;b+k,s)>0.$$

By Lemma 3 (iv), $\Delta^l g(b+n-l)=\Delta^l G(b,\alpha;b+n-l,s)>0$, and Lemma 4 is established for $i\in\Omega,\ S(i)=(n-k)(n-k-1)/2$.

Now, let, $j\in\Omega$. Assume that the assertion of Lemma 4 is valid for all $i\in\Omega, S(i)<S(j)$. Let $l\in\{0,\dots,n-1\}\setminus\{j_1,\dots,j_{n-k}\}$. First, consider the case, $l<j_{n-k}$. Choose the minimum number $q\in\{1,\dots,n-k\}$ such that $l<j_q$. Let $i\in\Omega$ be such that $i_p=j_p,\ p\neq q$, and $i_q=l$. Hence, $S(i)<S(j)$ and the inductive hypothesis applies to i. Fix $s\in I_b$ and consider $g(m)=G(b,j;m,s)-G(b,i;m,s)$. Then $Pg=0,\ m\in I_b$ and

$$\Delta^p g(a)=0,\qquad p=0,\dots,k-1;$$
$$\Delta^p g(b+n-p)=0,\qquad p\in\{j_1,\dots,j_{n-k}\}\setminus\{j_q\},$$
$$\Delta^{j_q}g(b+n-j_q)=-\Delta^{j_q}G(b,i;b+n-j_q,s).$$

By the inductive assumption, $\operatorname{sgn}\Delta^{j_q}g(b+n-j_q)$ is opposite $\operatorname{sgn}(-1)^{s(i,j_q)}$. By Lemma 3, $\operatorname{sgn}\Delta^l g(b+n-l)=\operatorname{sgn}\Delta^{j_q}g(b+n-j_q)$; since $\Delta^l g(b+n-l)=\Delta^l G(b,j;b+n-l,s)$, $\operatorname{sgn}(-1)^{s(j,l)}$ is opposite $\operatorname{sgn}(-1)^{s(i,j_q)}$ and the assertion is obtained in the case $l<j_{n-k}$.

As a last case, consider the case $l>j_{n-k}$. Set $i=(j_1,\dots,j_{n-k},l)$. Set $g(m)=G(b,j;m,s)-G(b,i;m,s)$. Employ the inductive hypothesis on k, and the argument proceeds as in the preceding cases.

THEOREM 5. *Assume that $Pu(m)=0,\ m\in I_\infty$, is right disfocal on I_∞. Let $i\in\Omega$, and assume that $i_{n-k}<n-1$. Let $\alpha>0$ be such that $i_\alpha=i_1+(\alpha-1),\ i_{\alpha+1}>i_1+\alpha$. Let $b_l\in I_\infty,\ l=1,2$, and assume $a+k\le b_1<b_2$. Then*

$$\left(-1\right)^{n-k}\Delta^{i_1}G(b_2,i;m,s)>\left(-1\right)^{n-k}\Delta^{i_1}G(b_1,i;m,s),\tag{6}$$

where (6) holds for $(m,s)\in\{a+k-i_1,\dots,b_1+n-i_1\}\times I_{b_1}$ if $b_2-b_1\ge\alpha$, or (6) holds for $(m,s)\in\{a+k-i_1,\dots,b_2+n-i_1-\alpha\}\times I_{b_1}$.

PROOF. Define $H(b,i;m,s)=G(b+1,i;m,s)-G(b,i;m,s)$. We shall show that

$$(-1)^{n-k}\Delta^{i_1}H(b,i;m,s)>0,\qquad \text{for } (m,s)\in\left\{a+k-i_1,\dots,b+n-i_1-\alpha\right\}\times I_b,$$

where $\alpha>0$ is such that $i_\alpha=i_1+(\alpha-1), i_{\alpha+1}>i_1+\alpha$. (6) follows immediately.

$\Delta^l H(b,i;b+n-l,s)=\Delta^l G(b+1,i;b+n-l,s)-\Delta^l G(b,i;b+n-l,s)$. If $l=i_j,\ l+1=i_{j+1}$ for some j, then $\Delta^l H(b,i;b+n-l,s)=0$. If $l=i_j,\ l+1<i_{j+1}$ for some j, or if $l=i_{n-k}$, then $\Delta^l H(b,i;b+n-l,s)=\Delta^l G(b+1,i;b+n-l,s)=-\Delta^{l+1}G(b+1,i;b+1+n-l-1,s)$. In particular, H, as a function of m, satisfies $PH(m)=0,\ m\in I_b$,

$$\Delta^l H(a)=0,\qquad l=0,\dots,k-1,$$
$$\Delta^l H(b+n-l)=-\Delta^{l+1}G(b+1,i;b+1+n-l-1,s),\qquad l=i_1,\dots,i_{n-k}.$$

It follows from Lemmas 3 and 4, see details provided in [8, Lemma 3.7], that

$$(-1)^{n-k}\Delta^{i_1}H(b,i;m,s)>0,\qquad \text{for } (m,s)\in\left\{a+k-i_1,\dots,b+n-i_1-\alpha\right\}\times I_b,$$

where $\alpha>0$ is such that $i_\alpha=i_1+(\alpha-1), i_{\alpha+1}>i_1+\alpha$.

To continue the development, set $\alpha = (0, \dots, n-k-1)$ and $\beta = (k, \dots, n-1)$. Next, we employ condition (4) and show that for $\beta = (k, k+1, \dots, n-1)$, the dominating function, $(-1)^{n-k} G(b, \beta; m, s)$ is monotone decreasing in b.

THEOREM 6. *Assume that $Pu(m) = 0$, $m \in I_\infty$, is right disfocal on I_∞. Let $\beta = (k, k+1, \dots, n-1)$. Let $b_l \in I_\infty, l = 1, 2$, and assume $a + k \le b_1 < b_2$. Then*

$$0 < (-1)^{n-k} \Delta^{k-1} G(b_2, \beta; m, s) < (-1)^{n-k} \Delta^{k-1} G(b_1, \beta; m, s), \tag{7}$$

$(m, s) \in \{a+1, \dots, b_1 + n - k + 1\} \times I_{b_1}$.

PROOF. Define H as in the proof of Theorem 5. Then $PH = 0, m \in I_b$, and

$$\Delta^l u(a) = 0, \qquad l = 0, \dots, k-1;$$
$$\Delta^l u(b + n - l) = 0, \qquad l = k, \dots, n-2,$$
$$\Delta^{n-1} u(b+1) = -\Delta^n G(b+1, \beta; b+1, s) \ge 0.$$

Note that condition (4) has been applied to obtain the sign of $\Delta^{n-1}u(b+1)$. Apply Lemma 3 and obtain that $(-1)^{n-k} \Delta^{k-1} H(b, \beta; m, s) < 0$ for $(m, s) \in \{a+1, \dots, b+n-k+1\} \times I_b$.

REMARKS.

(i) The inequalities, (5), (6), and (7) remain valid for Δ^l, $l = 0, \dots, i_1$, by direct summation and the boundary conditions, (3(b,i)).

(ii) Suppose the inequalities shown in (4) are reversed. Then $(-1)^{n-k} \Delta^{k-1} H(b, i; m, s) > 0$, for $(m, s) \in \{a+1, \dots, b+n-k+1\} \times I_b$. This is the setting where Elias [7] obtains his results for a unique limiting Green's function for the case of ordinary differential equations.

(iii) If $q_l \equiv 0$, $l = 0, \dots, n-1$, in (1), the proof of Theorem 6 readily shows that $G(b, \beta; m, s)$ is independent of b. See [9] for further discussion along these lines.

It is now an easy consequence of Theorems 2, 5 and 6 to see that $\lim_{b \to \infty} G(b, \gamma; m, s)$ is well-defined on $I_\infty \times I_\infty$ for $\gamma = \alpha$ or β.

THEOREM 7. *Assume that $Pu(m) = 0$, $m \in I_\infty$, is right disfocal on I_∞. Let $\gamma = \alpha$ or β. There exists $\mathbb{G}(\gamma; m, s)$ defined on $I_\infty \times I_\infty$ such that*

$$\lim_{b \to \infty} G(b, \gamma; m, s) = \mathbb{G}(\gamma; m, s), \qquad (m, s) \in I_\infty \times I_\infty.$$

Moreover, $(-1)^{n-k} \Delta^{k-1} \mathbb{G}(\beta; m, s) > 0$ on $\{a+1, \dots\} \times I_\infty$.

PROOF. Let $\gamma = \beta$. Let $(m, s) \in \{a+1, \dots\} \times I_\infty$. Let $\{b_l\} \subset \{m, m+1, \dots\}$ be such that $b_l \to \infty$ as $l \to \infty$. By Theorem 6, $(-1)^{n-k} \Delta^{k-1} G(b_l, \beta; m, s)$ is monotonically decreasing; by Theorems 1 and 2, the sequence is bounded below by 0. Hence, the sequence converges and $\Delta^{k-1} \mathbb{G}(\beta; m, s)$ is well-defined and of known sign. Apply the boundary conditions at a and direct summation to see that $\mathbb{G}(\beta; m, s)$ is well-defined and of known sign. Similarly, $\mathbb{G}(\alpha; m, s)$ is well-defined and of known sign; $(-1)^{n-k} G(b_l, \alpha; m, s)$ is monotone increasing and bounded above by $\mathbb{G}(\beta; m, s)$.

We now conclude this paper by arguing that $\mathbb{G}(\alpha; m, s) = \mathbb{G}(\beta; m, s) = \mathbb{G}(m, s)$, for $(m, s) \in I_\infty \times I_\infty$. It will then follow by Theorem 2 that if $i \in \Omega$, then

$$\lim_{b \to \infty} G(b, i; m, s) = \mathbb{G}(m, s), \qquad (m, s) \in I_\infty \times I_\infty. \tag{8}$$

We begin by stating three elementary lemmas.

LEMMA 8. *Let $u(m)$, $m = a$, $a+1, \dots$, be a sequence of reals and assume $\lim_{m \to \infty} u(m)$ exists. Then $\lim_{m \to \infty} \Delta u(m) = 0$.*

PROOF. Assume $\lim_{m\to\infty} u(m) = L$. The result follows since $|\Delta u(m)| \le |u(m)-L|+|u(m+1)-L|$.

REMARK. It is interesting to note that in the case for differential equations, if one assumes $\lim_{x\to\infty} y(x)$ exists, this is not sufficient to imply $\lim_{x\to\infty} y'(x) = 0$. By the Landau inequality, $\|y'\| \le 4\|y\|\,\|y''\|$, if one assumes, in addition that y'' is bounded, then in fact, $\lim_{x\to\infty} y'(x) = 0$. See [10, pp. 141].

LEMMA 9. *Let $u(m)$, $m = a,\ a+1,\dots$, be a sequence of reals and assume that there exists $M > a$ such that if $m \ge M$, then $\Delta u(m) \ge \gamma > 0$, for some real number, γ. Then $u(m) \to \infty$ as $m \to \infty$.*

PROOF. Let $m \ge M$. Then $u(m+k) \ge k\gamma + u(m)$, for each $k = 1, 2, \dots$.

LEMMA 10. *Assume $u(m)$, $m = a,\ a+1,\dots$, is a sequence of reals satisfying*

$$\begin{aligned} \Delta^n u(m) &\le 0, \\ \Delta^l u(a) &= 0, \qquad l = 0,\dots,k-1, \\ (-1)^{n-k}\Delta^k u(a) &\ge 0, \end{aligned} \tag{9}$$

$$\lim_{m\to\infty} \Delta^l u(m) = 0, \qquad l = k,\dots,n-1. \tag{10}$$

Then $u \equiv 0$.

PROOF. Since $\Delta^n u(m) \le 0$, $\Delta^{n-1}u \downarrow$. Given the boundary condition (10) at ∞, $\Delta^{n-1}u(m) \ge 0$. Repeating this argument, we obtain, $\Delta^{n-2}u(m) \uparrow$, and $\Delta^{n-2}u(m) \le 0$. Repeating the argument inductively, we obtain, $(-1)^{n-k}\Delta^k u(m) \uparrow$, and $(-1)^{n-k}\Delta^k u(m) \le 0$. Now apply the initial condition, $(-1)^{n-k}\Delta^k u(a) \ge 0$, and obtain $\Delta^k u \equiv 0$. Finally, apply the initial conditions in (9), and employ direct summation to complete the proof.

We shall now state and prove the main results in this paper in Theorem 11 and Corollary 12.

THEOREM 11. *Assume $Pu(m) = 0$ is right disfocal on I_∞ and assume (4) holds. Then for each $(m,s) \in I_\infty \times I_\infty$, $\mathbb{G}(\beta;m,s) = \mathbb{G}(\alpha;m,s)$.*

PROOF. Define $\mathbb{H}(m,s) = \mathbb{G}(\beta;m,s) - \mathbb{G}(\alpha;m,s)$. The proof is complete once we show that $\mathbb{H}$, as a function of m, satisfies the BVP, (9), (10). Note that since (4) holds, $\mathbb{H}$ satisfies (9) by Theorem 2. Hence, the proof reduces to showing that $\mathbb{H}$ satisfies (10). We do so by showing that each of $\mathbb{G}(\beta;m,s)$ and $\mathbb{G}(\alpha;m,s)$ satisfy (10).

First, consider $\mathbb{G}(\beta;m,s)$. We show that $\lim_{m\to\infty} \Delta^{k-1}\mathbb{G}(\beta;m,s)$ exists. It will then follow by Lemma 7 that $\mathbb{G}(\beta;m,s)$ satisfies (10). Recall [1,4] that $G(b,\beta;m,s)$ satisfies $Pu(m) = \delta_{m,s}$. Thus, for $m > s$, $\Delta^n G(b,\beta;m,s) \le 0$. Since $\Delta^{n-1}G(b,\beta;b,s) = 0$, it follows that for $m > s$, $\Delta^{n-1}G(b,\beta;m,s) \ge 0$, and so, for $m > s$, $\Delta^{n-1}\mathbb{G}(\beta;m,s) \ge 0$. Repeat this argument inductively and obtain $(-1)^{n-k}\Delta^k\mathbb{G}(\beta;m,s) \le 0$ for $m > s$. By Theorems 1 and 2, $(-1)^{n-k}$ $\Delta^{k-1}\mathbb{G}(\beta;m,s) \ge 0$ for all m and so, $\lim_{m\to\infty} \Delta^{k-1}\mathbb{G}(\beta;m,s)$ exists, since as a function of m, $(-1)^{n-k}\Delta^{k-1}\mathbb{G}(\beta;m,s)$ is a nonnegative decreasing function. Thus, $\mathbb{G}(\beta;m,s)$ satisfies (10).

Second, consider $\mathbb{G}(\alpha;m,s)$. It follows as seen in the preceding paragraph, that for $m > s$, $\Delta^n G(b,\alpha;m,s) \le 0$; hence, for $m > s$, $\Delta^n\mathbb{G}(\alpha;m,s) \le 0$. It follows that, $\lim_{m\to\infty} \Delta^{n-1}\mathbb{G}(\alpha;m,s)$ exists or diverges to $-\infty$. Assume for the sake of contradiction that

$$(-1)^{n-k} \lim_{m\to\infty} \Delta^{n-1}\mathbb{G}(\alpha;m,s) < 0,$$

where we include the divergent case, if appropriate. Then, by repeated applications of Lemma 8, $(-1)^{n-k}\mathbb{G}(\alpha;m,s) < 0$, eventually. This contradicts Theorems 1 and 2. Now, assume for the sake of contradiction that $(-1)^{n-k} \lim_{m\to\infty} \Delta^{n-1}\mathbb{G}(\alpha;m,s) > 0$. Then

$$(-1)^{n-k} \lim_{m\to\infty} \Delta^{n-1}\mathbb{H}(m,s) < 0.$$

It follows by repeated applications of Lemma 8 that eventually, $(-1)^{n-k}\mathbb{H}(m,s) < 0$. This contradicts Theorem 2. Thus, $\lim_{m\to\infty} \Delta^{n-1}\mathbb{G}(\alpha; m, s) = 0$.

As noted in the previous paragraph, $\Delta^{n-1}\mathbb{G}(\alpha; m, s) \downarrow$ in m, for $m > s$. We have just shown that $\lim_{m\to\infty} \Delta^{n-1}\mathbb{G}(\alpha; m, s) = 0$; thus, $\Delta^{n-1}\mathbb{G}(\alpha; m, s) \geq 0$, for $m > s$. The preceding argument now applies and it follows that $\lim_{m\to\infty} \Delta^{n-2}\mathbb{G}(\alpha; m, s) = 0$. It now follows inductively that for $l = k, \ldots, n-1$, $\lim_{m\to\infty} \Delta^{l}\mathbb{G}(\alpha; m, s) = 0$, and the proof of Theorem 10 is complete.

COROLLARY 12. *Assume the hypotheses of Theorem 11. Let $i \in \Omega$. Then* $\lim_{b\to\infty} G(b, i; m, s) = \mathbb{G}(m, s)$.

PROOF. This is an immediate consequence of Theorems 2 and 11.

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